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LOW-FREQUENCY RESONANCES IN THE DYNAMIC INTERACTION OF AN ELASTIC SOLID WITH A SEMI-BOUNDED MEDIUM†

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The vibration of an elastic rod with a large point mass at one end to which a given harmonic load is applied is investigated. The other end of the rod is connected to a rigid punch in continuous contact with a semi-bounded elastic medium. Some lemmas are proved on systems with simplified boundary conditions with properties which majorize the characteristics of the initial system. The effect of the system parameters on the conditions under which an unbounded resonances occur is thereby determined. Analytic relations for determining the number of resonance frequences are obtained. © 1999 Elsevier Science Ltd. All rights reserved.

It has been established [1] that unbounded low-frequency resonances occur in the interaction of a massive body with a semi-bounded medium. Under certain conditions this can happen when elastic solids of finite dimensions are in contact with an elastic base, as noted in [2] when investigating the flexural vibration of an elastic beam on the layer surface. The model of a bounded body considered below gives a clear picture of the mechanical nature of the occurrence of resonances when an elastic solid of finite size interacts with a semi-bounded medium.

1. We consider an elastic two-mass system (referred to below as "the system"), which contains an elastic rod which connects a massive body M_1 with a rigid punch M_2 occupying a region Ω on the surface of the medium. The base is a semi-bounded medium (the 'medium') with a critical frequency of wave propagation (a layer, a packet of layers, etc.). The system vibrates vertically due to a harmonic load applied to the body M_1 (the time dependence is taken in the form $\exp(-i\omega t)$). There is zero friction in the contact area.

The investigation of the resonance properties of the system reduces to a boundary-value problem which, in dimensionless variables (the time factor is omitted here and below), has the form

$$w_{xx} = -\sigma^2 w \tag{1.1}$$

$$x = l: -m_1\kappa_2^2 w = F - ESw_x \tag{1.2}$$

$$x = 0: -m_2\kappa_2^2 w = ESw_x - Pw, \quad P = \iint\limits_{\Omega} q(x_1, x_2)dx_1dx_2 \tag{1.3}$$

Here F is the amplitude of the external load, $\kappa_2^2 = \rho\mu^{-1}a^2\omega^2$ is the vibration frequency relative to the parameters of the medium, ρ and μ are the density and modulus of elasticity of the material of the medium, $\sigma = \sigma_0\kappa_2$ is the relative frequency of vibration of the rod, where $\sigma_0^2 = \rho_0 E^{-1}$, E, ρ_0 are Young's modulus an the density of the material of the rod relative to the respective parameters of the material of the medium, m_n ($n = 1, 2$) is the mass of the body M_n and $q(x_1, x_2)$ are the stresses in the contact area, which satisfy the integral equation

$$kq = \iint\limits_{\Omega} k(x_1 - \xi, x_2 - \eta)q(\xi, \eta)d\xi d\eta = 1, \ (x_1, x_2) \in \Omega \tag{1.4}$$

$$k(x_1, x_2) = \int\limits_{\Gamma_1}\int\limits_{\Gamma_2} K(\alpha, \beta)e^{-i(\alpha x_1 + \beta x_2)}d\alpha d\beta$$

The form of the functions $K(\alpha, \beta)$, the rule used to choose contours Γ_1 and Γ_2 for different contact problems, the methods of solving integral equations of the type (1.4), and the properties of these solutions, are given in [3, 4]. $P(\kappa_2)$, the reaction of the base to a unit displacement of the punch, is a

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real function in the frequency range $[0, \kappa^*]$ and complex-valued outside that range; κ^* is the first critical frequency of wave propagation. It is this which governs the features of resonance interaction of bounded elastic bodies with semi-bounded media. In particular, the value of the critical frequency determines the boundary of the region of existence of unbounded resonances. We will refer to boundary-value problem (1.1)–(1.3) as Problem 1.

We will now consider special cases of conditions (1.2) and (1.3)

$$x = l: F - ESw_x = 0 \tag{1.5}$$

$$x = 0: w = 0 \tag{1.6}$$

We will refer to boundary-value problem (1.1), (1.2), (1.6) as Problem 2, and to problem (1.1), (1.5), (1.6) as Problem 3. In terms of mechanics, Problems 2 and 3 describe the vibrations of a "clamped" system or "clamped" rod, respectively.

The solution of Problem 1 has the form (w_n is the amplitude of the vibration of the solid M_n)

$$w_1 = F[\gamma_0\gamma_1 + E\sigma]\Delta_0^{-1}, \quad w_2 = FE\sigma(\cos\sigma l\Delta_0)^{-1} \tag{1.7}$$

where

$$\gamma_0 = \mathrm{tg}\sigma l, \quad \gamma_1 = P - m_2\kappa_2^2, \quad \gamma_2 = E\sigma(\gamma_0 + \gamma_3^{-1})\Delta_1^{-1}, \quad \gamma_3^{-1} = m_1^0\sigma \tag{1.8}$$

$$\Delta_0 = E\sigma\Delta_1(\gamma_1 - \gamma_2), \quad \Delta_1 = 1 - \gamma_0\gamma_3^{-1}, \quad m_1^0 = m_1\rho_0^{-1}$$

It follows from (1.7) and (1.8) that the resonance frequencies of a system which interacts with an elastic base are the eigenvalues (EV) κ_n^0 of Problem 1, which satisfy the equation

$$\Delta_0(\kappa_2) = 0 \tag{1.9}$$

If κ_n^0 are real, the system in contact with an elastic base has an unbounded resonance.

Lemma 1. The real roots κ_n^0 of Eq. (1.9) satisfy the inequality $\kappa_n^0 < \kappa^*$.

The lemma follows from the fact that Eq. (1.9) cannot have real roots for complex values of $P(\kappa_2)$.

The EV z_n of Problem 2 satisfy the equation

$$\Delta_1(\kappa_2) = 0 \tag{1.10}$$

The EV z_n^0 of Problem 3 are the roots of the equation $\cos\sigma l = 0$. By virtue of the properties of Problems 2 and 3, z_n and z_n^0 are real, with $z_n^0 = (n - 1/2)\pi\sigma_0^{-1}l^{-1}$ ($n = 1, 2, 3, \ldots$).

2. Let N be the number of EV z_n and N_0 the number of EV z_n^0 in the interval $[0, \kappa^*]$, that is, $z_N \leqslant \kappa^*$ and $z_{N+1} > \kappa^*$, $z_{N_0+1}^0 \leqslant \kappa^*$ and $z_{N_0+1}^0 > \kappa^*$, where $N_0 = [\kappa^*\pi^{-1}\sigma_0 l + 1/2]$ ($[a]$ is the integer part of a).

Note that $\kappa^* = \pi/2$ [3, 4] in the case of an elastic layer with a clamped lower face, that is

$$N_0 = [(\sigma_0 l + 1)/2] \tag{2.1}$$

Theorem. The number of unbounded resonances κ_n^0 of a system in contact with an elastic medium N^* satisfies the inequality $N^* \geqslant N_0$, where $N^* = N_0 + 1$ if one of the inequalities $\gamma_1(\kappa^*) \leqslant \gamma_2(\kappa^*)$ or $\gamma_0(\kappa^*) \geqslant \gamma_3(\kappa^*)$ is satisfied, $N^* = N_0 + 2$ if both are satisfied, and $N^* = N_0$ if neither is satisfied.

To prove the theorem, we will establish a relation between the EV of boundary-value problems 1, 2 and 3 and the number that there are.

Lemma 2. For the EV of Problems 1 and $\kappa_n^0 \leqslant z_n$ in the interval $[0, \kappa^*]$.

It follows from (1.8) that z_n are poles of the function $\gamma_2(\kappa_2)$ (curve 2 of Fig. 1). The lemma follows from the continuity of $\gamma_2(\kappa_2)$ in the interval $[z_{k-1}, z_k]$, ($k = 1, 2, 3, \ldots; z_0 = 0$) and the boundedness of $\gamma_1(\kappa_2)$ (curve 1 of Fig. 1).

Lemma 3. The EV of Problems 2 and 3 satisfy the inequality $x_n \leqslant z_n^0$.

To prove the lemma, we convert (1.10) to the form

$$\gamma_0(\kappa_2) - \gamma_3(\kappa_2) = 0$$

According to (1.8) z_n^0 are poles of the function $\gamma_0(\kappa_2)$ (curve 2 of Fig. 2). The lemma follows from the behaviour of the functions $\gamma_0(\kappa_2)$ and $\gamma_3(\kappa_2)$ (curve 1 of Fig. 2) in the interval $[z_{k-1}^0, z_k^0]$ ($k = 1, 2, 3, \ldots$; $z_0^0 = 0$).

It follows from Lemmas 2 and 3 that, just as when a "clamped" rod is loaded, replacing a rigid base by an elastic base reduces its resonance frequencies: $\kappa_n^0 \leqslant z_n \leqslant z_n^0$.

The relation between the number of EV of Problems 1, 2 and 3 in the interval $[0, \kappa^*]$ can be established by the following lemmas.

Lemma 4. If $\gamma_1(\kappa^*) \leqslant \gamma_2(\kappa^*)$, then $N^* = N + 1$, and otherwise $N^* = N$.

According to Lemma 2 $\kappa_N^0 \in [0, z_N]$. It follows from the behaviour of $\gamma_1(\kappa_2)$ and $\gamma_2(\kappa_2)$ in the interval $[z_N, \kappa^*]$ that $\kappa_{N+1}^0 \in [z_N, \kappa^*]$, where $\gamma_1(\kappa^*) \leqslant \gamma_2(\kappa^*)$.

Lemma 5. If $\gamma_3(\kappa^*) \leqslant \gamma_0(\kappa^*)$, then $N = N_0 + 1$ and otherwise $N = N_0$.

By virtue of Lemma 3, $z_{N_0} \in [0, z_{N_0}^0]$. Similarly $z_{N_0+1} \in = z_{N_0}^0, \kappa^*$ when $\gamma_0(\kappa^*) \geqslant \gamma_3(\kappa^*)$.

Lemma 5 completes the proof of the theorem.

The theorem gives a clear idea of the effect of the parameters of the system on the resonance conditions, from which the number of low-frequency unbounded resonances can be determined without having to make a complicated analysis of the dynamic contact of the system with an elastic base and, if necessary, can be altered by choosing the parameters of the system appropriately.

Corollary 1. When $m_1 = 0$, the number of unbounded resonances $N^* = N_0 + 1$, if $\gamma_1(\kappa^*) \leqslant \gamma_2(\kappa^*)$. Otherwise $N^* = N_0$.

The corollary follows from Theorem 1, since the EV of Problems 2 and 3 are equal when $m_1 = 0$.

Corollary 2. If $\sigma_0 l > 1/2$, a system in contact with an elastic medium has at least one unbounded resonance for a punch of any mass m_2.

Corollary 3. If $m_2 > P(\kappa^*)\kappa^{*-2}$, a system in contact with an elastic medium has at least one unbounded resonance, whatever the length of the rod.

Remark. The values of m_1 and m_2 affect not only the number of resonances, but also the values of the resonance frequencies, m_2 having a direct influence on κ_n^0 (Fig. 1), while m_1 has an indirect effect through the values z_n (Fig. 2).

1. For any values of m_2 we have $z_{n-1} < \kappa_n^0 < z_n$. An increase of m_2 leads to a reduction in κ_n^0. For small values of m_2, the effect of a change in these values on the resonance frequencies increases with the number of the resonance frequency. As m_2 increases (as κ_n^0 approaches z_{n-1}) the quantities κ_n^0 change more slowly, approaching a limit, first for higher and then for lower resonance frequencies. Ultimately, for sufficiently large values of m_2, any change in these values affects only the first resonance.

2. For any values of m_1 we have $\zeta_n^0 < z_n < z_n^0$, where ζ_n^0 are the zeros of the function $\gamma_0(\kappa_2)$. An increase in m_1 leads to a reduction in the values of z_n, which in turn reduces the resonance frequencies κ_n^0. The rate of

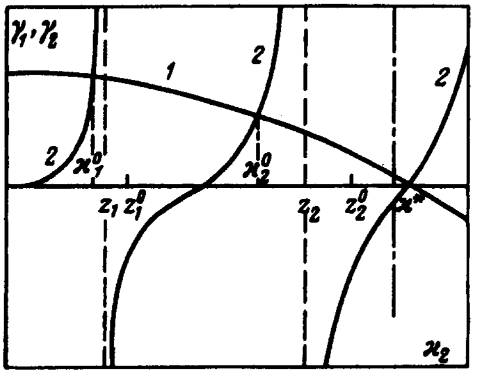

Fig. 1.

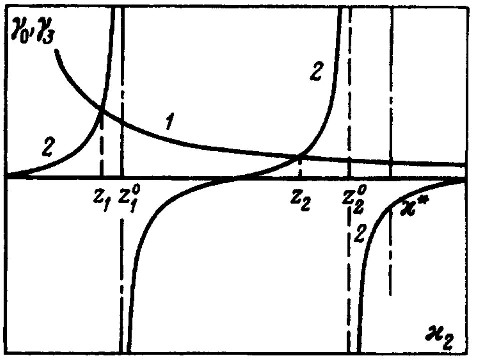

Fig. 2.

change of z_n for small m_1 increases with the number of the resonance frequency. As m_1 increases (as z_n approaches ζ_n^0) z_n changes more slowly, approaching a limit but not reaching ζ_n^0. As before, this occurs first for very high, and then for lower frequencies. Ultimately, if m_1 is large enough, any change in its value affects only the first resonance.

3. As an example, we will consider the plane problem of the vibration of an elastic two-mass system on the surface of an elastic layer of thickness h with clamped lower boundary. The system is a rigid strip-like punch M_2 which, in plane, occupies the region $|x_1| \leqslant a$, and is connected by a rod of finite length l to the massive body M_1. Young's modulus and density of the rod material are E and ρ_0, respectively. It is assumed that the punch, rod and body M_1 perform translational vertical vibration under the effect of a harmonic load applied to the body M_1. The boundary-value problem for the motion of the components has the same form as (1.1), (1.2), except that the reaction of the medium $P(\kappa_2)$ in this case is represented by the expression

$$P = \int_{-a}^{a} q(x_1)dx_1 \tag{3.1}$$

The contact stresses $q(x_1)$ satisfy the integral equation

$$\int_{-a}^{a} k_{33}(x_1 - \zeta)q(\zeta)d\zeta = 1, \quad k_{33}(s) = \int_{\Gamma_1} K_{33}(\alpha)e^{-i\alpha s}d\alpha \tag{3.2}$$

The form of the function $K_{33}(\alpha)$ for the problem of the vibration of a rigid punch on the surface of a layer with a clamped base is well known and will not be given here. We use the factorization method [3, 4] to solve integral equation (3.2). The function $q(x_1)$ and the reaction of the base (3.1) are given by well-known formulae [3–6].

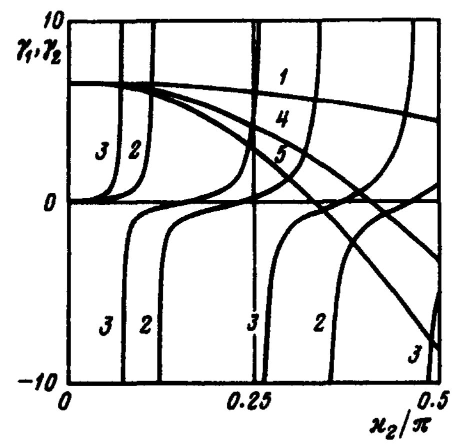

Fig. 3.

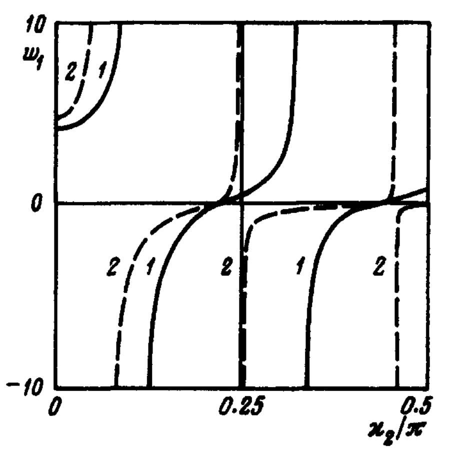

Fig. 4.

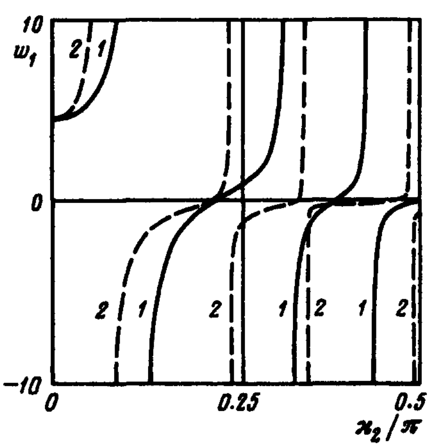

Fig. 5.

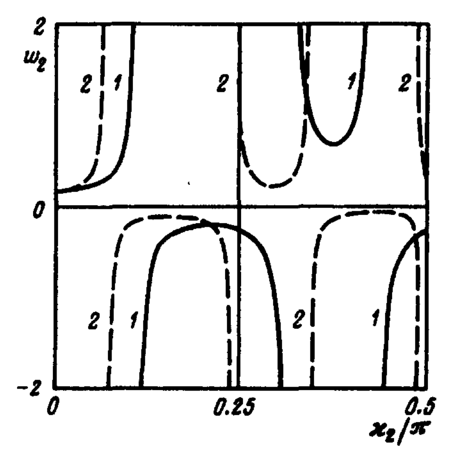

Fig. 6.

Figure 3 shows graphs of the functions $\gamma_1(\kappa_2)$ and $\gamma_2(\kappa_2)$, illustrating the influence of the values of m_1 and m_2 on the resonance properties of an elastic two-mass system in contact with an elastic base. The parameters of the rod have values such that $\sigma_0 l = 4.32$. In this case from (2.1) we have $N_0 = 2$. The numbers 1, 4 and 5 refer to the curves $\gamma_1(\kappa_2)$ with $m_2 = 0, 3, 5$ and the numbers 2 and 3 refer to curves $\gamma_2(\kappa_2)$ for $m_1 = 0, 3$ respectively. Clearly, $\gamma_0(\kappa^*) \geqslant \gamma_3(\kappa^*)$ when $m_1 = 3$, since [3, 4] $\kappa^* = \pi/2$.

It follows from the graphs that the following situations can arise with the given values of the rod parameters.

Configuration a: $m_1 = 0, m_2 = 0$ (curves 1 and 2 of Fig. 3); Corollary 1 implies that the system has two resonances, since $\gamma_1(\kappa^*) > \gamma_2(\kappa^*)$;

Configuration b: $m_1 = 0, m_2 = 3$ (curves 4 and 2 of Fig. 3); Corollary 1 implies that the system has three resonances, since $\gamma_1(\kappa^*) \leqslant \gamma_2(\kappa^*)$;

Configuration c: $m_1 = 3, m_2 = 0$ (curves 1 and 3 of Fig. 3); it follows from the theorem that the system has three resonances, since $\gamma_1(\kappa^*) \leqslant \gamma_2(\kappa^*)$ and $\gamma_0(\kappa^*) \geqslant \gamma_3(\kappa^*)$;

Configuration d: $m_1 = 3, m_2 = 5$ (curves 5 and 3 of Fig. 3); it follows from the theorem that the system has four resonances, since $\gamma_1(\kappa^*) \leqslant \gamma_2(\kappa^*)$ and $\gamma_0(\kappa^*) \leqslant \gamma_3(\kappa^*)$.

Figures 4 and 5 show graphs of displacements w_1 as a function of the frequency. The numbers 1 and 2 in Fig. 4 refer to the curves of w_1, corresponding to configurations a and c of the system and the numbers 1 and 2 in Fig. 5 refer to the curves of w_1, corresponding to configurations b and d of the system. The system with configuration a has only two resonance frequencies for a given value of $\sigma_0 l$ (curve 1 of Fig. 4). The values of the resonance frequencies decrease as m_1 increases. When $\sigma_0 l$ (curve 1 of Fig. 4). The values of the resonance frequencies decrease as m_1 increases. When $m_1 = 3$ (configuration c, curve 2 of Fig. 4) the system has three isolated resonances. It was founded from the calculations that any further increase in m_1 leads to a decrease in the values of the resonance frequencies, the Q-factor of the system is improved (a considerable increase in the amplitude of oscillations is observed in a small neighbourhood of the resonance frequency), but the number of resonances is unaltered.

Comparing curves 1 of Figs 4 and 5, we see that an increase in m_2 displaces the resonances to lower frequencies. When $m_2 = 3$ (configuration b) a third resonance appears. The calculations show that in this case a further increase in m_2 merely displaces the resonances to lower frequencies, there is little change in the Q-factor of the system and the number of resonances remains the same.

Comparing curves 2 of Figs 4 and 5, we see that an increase in m_2 (configuration d) displaces the resonances to lower frequencies, and a fourth resonance appears. Any further increase in m_2 has no effect on the number of resonances.

Figure 6 shows curves of w_2, illustrating the displacement of the punch as a function of frequency. The numbers 1 and 2 refer to configurations b and d of the system. We see that w_2 behaves in a similar way to w_1, except that there are no "anti-resonance" frequencies, or frequencies at which the amplitude of the oscillations becomes zero.

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